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On the application of difference potential theory to active noise control[☆]

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Abstract

The application of the theory of difference potentials to the problem of active shielding and noise control is considered. Difference potential theory allows us to obtain the general solution of the problem in a finite-difference formulation. The solution is valid for arbitrary space domains, medium and boundary conditions. It only requires the information on the total sound (both “friendly” and “adverse”) at the perimeter of the domain to be shielded. In contrast to the previous publications, in the current paper the mechanism of active shielding solution based on the difference potential theory is analysed. The theory of difference potentials is applied to the system of acoustic equations. The correspondence between the finite-difference solution and the continuous solution based on Green's function is shown for the case of a uniform medium. Different possible representations of the active shielding source terms are analysed. A clear physical interpretation of the optimal space step in the finite-difference solution is provided. The results can be important for both understanding the solution of the active shielding problem and practical applications.

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1. Introduction

The problem of active noise control is a relatively new but extensively developed research activity in acoustics. In this problem, it is assumed that either some internal or external area is to be shielded via implementation of additional (secondary) sources, i.e. active shielding (AS). This is a key distinguishing feature of the approach from “passive” shielding where noise reduction is performed via mechanical means. Obviously, the mechanical route to shielding cannot be always implemented. In fact, often active and passive controls should be combined. The latter control may not be sufficient to filter low frequencies, while the former control is mostly appropriate for such frequencies. The problem becomes much more complicated however, if some “friendly” sound is assumed to be in the shielded area.

There are many publications devoted to the problem of AS and sound control. First theoretical papers in this field by Jessel, Malyuzhinets and Fedoryuk appeared about 30 years ago [4,7,15], while the first publications related to realistic practical applications appeared much later (see, e.g., [1,3]). Some of the noise suppression techniques are based on sound control in selected discrete [1–3,26] or directional [27] areas. Other techniques, in particular those developed by Kincaid et al. [9,10], assume detailed information regarding the sources and nature of noise. A number of publications are devoted to optimization of the strengths of spatially distributed secondary sources to minimize a quadratic pressure cost function [17,19]. The JMC method [8, 16,25], based on Huygens’ principle, requires only the information on the undesirable field in the perimeter of the shielded domain. Yet this method cannot be used if a desirable field, generated in the shielded domain, has to be taken into account. The most comprehensive theoretical and practical reviews of the AS problem and its technical implementation can be found in books [5, 18,23] and report [25].

As mentioned above, generally in the standard approaches to AS, it is necessary to know the characteristics of “adverse” sources including their location. From a practical standpoint, this information is not often available. A separate class of methods requires the information on total sound (both “friendly” and “adverse”) only at the perimeter of the domain to be shielded. It is very important to emphasize that the knowledge of both the “adverse” and “friendly” components is not necessarily required. Generally, these approaches are based on accessibility of Green’s function [15,24] where the exact solution of the AS problem is obtained for the Helmholtz equation with constant coefficients. An original approach based on the Difference Potential Theory (DPT) [20,21] allows the ability to achieve the same end in a much more general formulation. This solution is applicable to arbitrary geometric configurations, medium and boundary conditions. There are only two principal requirements for its practical implementation. The problem must be linear and possess a unique solution. In contrast to the other approaches described above, the ultimate AS solution is achieved in a finite-difference form. From a practical standpoint this may not necessarily be treated as a drawback because the implementation of the AS assumes some discrete distribution of the AS sources. This approach has been analysed for application to the Helmholtz equation and its analogue with variable coefficients in [11–14]. A comprehensive analysis of continuous and finite-difference surface potentials mostly appropriate for the Helmholtz-type equation is carried out by Tsynkov in [24]. In [22] the solution of the linear AS problem is obtained in the continuous space for the acoustic equation with constant and variable coefficients. Whereas optimization of AS sources is studied in detail in the papers cited above, not much attention has been paid to the wave analysis of AS process itself.

It is to be noted that the general solution of the AS problem is generally applicable to 3D problems. Yet some important properties of the solution remain hidden unclear behind the math-

emational formalism. In the current paper the details of the AS solution based on the DPT are provided for the case of monochromatic wave propagation in a duct with a termination. The technique [21] is subsequently applied to the system of acoustic equations. It is shown how waves, that are formally identical, can be distinguished in the AS solution. It is proved that the AS sources are uniquely defined under some general conditions; otherwise, the solution becomes non-unique, and some optimal solution is obtained. The AS source term is derived in a one-layer form, and its physical interpretation is given. The correspondence between the finite-difference and continuous solutions is shown. It is demonstrated that the knowledge of the reflection coefficients of the duct sides is not required for the AS provided by the DPT. This property of the AS solution may have a substantial impact on its potential applications because in practical problems the values of reflection coefficients are usually not known.

2. Mathematical statement of the active shielding problem

Mathematical formulation of the AS problem is presented in the following form. Let us assume that noise propagation is described by some linear boundary value problem in a domain D_0 :

$$Lw = S, \quad (1)$$

$$w \in U_{D_0}, \quad (2)$$

where U_{D_0} is a linear subspace of functions defined on $\overline{D_0}$ such that the solution of problem (1), (2) exists and is unique.

In particular, the domain D_0 may be a free space, and the boundary condition (2) can be represented by the Sommerfeld-type boundary conditions [11].

Let us consider now some domain D such that $D \subset D_0$. The sources on the right-hand side can be situated both in D and outside of D :

$$\begin{aligned} S &= S_f + S_a, \\ \text{supp } S_f &\subset D, \\ \text{supp } S_a &\subset D_0 \setminus D. \end{aligned} \quad (3)$$

Here, S_f is the source of “friendly” sound, while S_a is the source of “adverse” noise.

Assume that we know the distribution of function $w_{\partial D}$ at the boundary of D . It is important to emphasize that only this information is assumed to be available. In particular, the distribution of the sources S on the right-hand side is not known. The AS problem is reduced to seeking additional sources g in $D_0 \setminus D$ such that the solution of problem

$$Lw = S + g, \quad (4)$$

$$w \in U_{D_0}, \quad (5)$$

coincides with the solution of problem (1), (2) on subdomain D with $S = S_f$. Thus, the AS problem mathematically belongs to the class of inverse source problems [6]. It is worth noting here that an “obvious” solution $g = -S_a$ is not appropriate because the distribution of S_a is not known. Moreover, if the density S_a is known, the trivial solution $g = -S_a$ does not appear to be realistic for a practical implementation.

3. Difference potential formalism and main theorem

Following the DPT [21], let us introduce some grid M^0 in D_0 . We introduce subsets of grid M^0 as follows: $M^+ = M^0 \cap \overline{D}$, $M^- = M^0 \setminus M^+$. Assume that Eq. (1) is approximated on some stencil by equation

$$L_h w^{(h)}|_m \stackrel{\text{def}}{=} \sum_n a_{mn} w_n^{(h)} = S_m^{(h)}, \quad m \in M^0. \quad (6)$$

Equation (6) is completed by the boundary condition approximating continuous boundary condition (2):

$$w^{(h)} \in U_{D_0}^{(h)}, \quad (7)$$

where $U_{D_0}^{(h)}$ is a linear discrete space that is a discrete counterpart of space U_{D_0} .

The extensions of sets M^0 , M^+ , M^- due to the stencil we denote as N^0 , N^+ , N^- , respectively. It is clear that the sets N^+ and N^- intersect each other. We interpret their intersection as the grid boundary γ of the domain D : $\gamma = N^+ \cap N^-$. In turn, the grid boundary γ is split into two nonintersecting sub-boundaries: $\gamma = \gamma^+ \cup \gamma^-$, where $\gamma^+ = \gamma \cap M^+$ and $\gamma^- = \gamma \cap M^-$. Now we seek the finite-difference solution of AS problem (4), (5).

The AS problem is then formulated in a finite-difference form as follows. We consider problem (6), (7) where

$$\begin{aligned} S_m^{(h)} &= S_{f|m}^{(h)} + S_{a|m}^{(h)}, \\ \text{supp } S_f^{(h)} &\subset M^+, \\ \text{supp } S_a^{(h)} &\subset M^-. \end{aligned} \quad (8)$$

It is required to find such an additional source term

$$g^{(h)}: \text{supp } g^{(h)} \subset M^- \quad (9)$$

that the solution of the problem

$$\begin{aligned} L_h w_g^{(h)}|_m &= S_m^{(h)} + g^{(h)}, \quad m \in M^0, \\ w_g^{(h)} &\in U_{D_0}^{(h)}, \end{aligned} \quad (10)$$

coincides on $N^+ \subset N^0$ with the solution of problem (6)–(8) if $S_a^{(h)} \equiv 0$:

$$\begin{aligned} L_h w_f^{(h)}|_m &= S_m^{(h)}, \quad m \in M^0, \\ w_f^{(h)} &\in U_{D_0}^{(h)}. \end{aligned} \quad (11)$$

The only function required, w_γ , is supposed to be known, say, from measurements.

The general solution of the AS problem in the discrete formulation is provided by the following primary theorem [20].

Theorem 1. *The general solution of AS problem (6), (7), (10), (11) is given by*

$$g^{(h)} = \begin{cases} -L_h v^{(h)}|_{M^-}, & \text{if } m \in M^-, \\ 0, & \text{if } m \in M^+, \end{cases} \quad (12)$$

where $v^{(h)}$ is an arbitrary function such that

$$v^{(h)} \in U_{D_0}^{(h)}, \quad v_\gamma^{(h)} = w_\gamma^{(h)}. \quad (13)$$

Thus, if the source term $g^{(h)}$ satisfies the conditions of the theorem, then $w_{g|N^+}^{(h)} = w_{f|N^+}^{(h)}$.

The proof of this theorem can be found in [21]. It is important to emphasize that solution (12) is obtained for the general statement of the problem formulated in the previous section and not based on the knowledge of the specific Green's function.

It is clear that the function $v^{(h)}$ in (12) is not unique. A partial case of this function corresponds to $v^{(h)}|_{M^0 \setminus \gamma} = 0$. In this case the AS source term is only placed on a minimal possible area, specifically at γ^- . In [11], it is formulated as a proposition that such sources are to be minimal in L_1 .

4. Discrete solution of 1D acoustic equations with termination

Currently, in the literature solution (12) has been only applied to the Helmholtz equation [11–14,24]. We implement this technique to the acoustic equations. It allows us to obtain some additional properties of the solution. In order to understand better the nature and characteristics of the solution, it is investigated in a 1D case with application to a duct flow. The primary sources are assumed to be in the interval of $0 < x \leq L$, while the secondary source is placed at $x = 0$ to shield the area $x < 0$. It is to be noted that the locations of the primary sources are unknown. The duct is assumed to be closed at $x = L$. For the sake of simplicity, temporarily, we assume that the duct is closed at $x = -L$ by an absolutely absorbing wall.

Considering the 1D system of acoustic equations for isentropic flow:

$$\begin{aligned} p_t + \rho c^2 u_x &= f_p, \\ u_t + p_x / \rho &= f_u. \end{aligned} \quad (14)$$

Here, p is the sound pressure, ρ is the density of air, u is the particle velocity and c is the sound speed, f_p and f_u are acoustic sources.

Assume that the acoustic sources are time-harmonic:

$$\begin{aligned} f_p &= \rho c^2 \hat{f}_p e^{i\omega t}, \\ f_u &= c \hat{f}_u e^{i\omega t}. \end{aligned} \quad (15)$$

Hence, the dependent variables can be rewritten in the following form:

$$\begin{aligned} p &= \rho c \hat{p} e^{i\omega t}, \\ u &= \hat{u} e^{i\omega t}. \end{aligned} \quad (16)$$

The equations for the Fourier spectrum are as follows:

$$\begin{aligned} ik\hat{p} + \hat{u}_x &= \hat{f}_p, \\ ik\hat{u} + \hat{p}_x &= \hat{f}_u. \end{aligned} \quad (17)$$

The system for the Fourier components can easily be written in the following characteristic form:

$$\begin{aligned} \widehat{L}^+ \widehat{R}^+ &= \hat{f}^+, \\ \widehat{L}^- \widehat{R}^- &= \hat{f}^-, \end{aligned} \quad (18)$$

where $\widehat{L}^+ \stackrel{\text{def}}{=} ik + \frac{\partial}{\partial x}$, $\widehat{L}^- \stackrel{\text{def}}{=} ik - \frac{\partial}{\partial x}$, $\widehat{R}^+ \stackrel{\text{def}}{=} \hat{p} + \hat{u}$, $\widehat{R}^- \stackrel{\text{def}}{=} \hat{p} - \hat{u}$, $\hat{f}^+ = \hat{f}_p + \hat{f}_u$, $\hat{f}^- = \hat{f}_p - \hat{f}_u$. The functions \widehat{R}^+ and \widehat{R}^- are the Fourier counterparts of the Riemann invariants of system (14) propagating along the characteristics $\frac{dx}{dt} = c$ and $\frac{dx}{dt} = -c$, respectively.

In order to consider a finite-difference formulation of the problem, let us introduce a uniform grid with a constant step $h = L/M$. Assume that the grid M^0 is represented by nodes $i = -M, \dots, M$, then the set M^+ corresponds to nodes $i = -M, \dots, -1$.

It is natural to approximate these equations on account of the hyperbolic properties of the original equations written in characteristics. It can be done if we use the “upwind” approximation:

$$\left(ik + \frac{1}{h} \nabla \right) \widehat{R}_m^+ = \hat{f}_m^+, \quad (19)$$

$$\left(ik - \frac{1}{h} \Delta \right) \widehat{R}_m^- = \hat{f}_m^-, \quad m \in M^0, \quad (20)$$

where

$$\nabla s_i \stackrel{\text{def}}{=} s_i - s_{i-1}, \quad \Delta s_i \stackrel{\text{def}}{=} s_{i+1} - s_i.$$

Then, in the case of Eq. (19) for \widehat{R}^+ the boundary $\gamma \equiv \gamma^+ = M^+ \cap N^-$ is one-layer and corresponds to $m = -1$, while $\gamma^- = \emptyset$. In turn, in the case of Eq. (20) the boundary $\gamma \equiv \gamma^-$ is also one-layer and coincides with the point $m = 0$.

System (19), (20) is completed by the following boundary conditions:

$$\widehat{R}_{-M}^+ = 0, \quad \widehat{R}_M^- = r_l \widehat{R}_M^+. \quad (21)$$

Here, r_l is the reflection coefficient of the wall on the right-hand side. The reflection coefficient $r_l = 0$ corresponds to a fully absorbing wall while $r_l = 1$ represents the case of an absolutely reflecting wall.

The solution of problem (19)–(21) is given by the following two propositions.

Proposition 1. *The solution of Eq. (19) with boundary condition (21) is as follows:*

$$\widehat{R}_m^+ = \sum_{p \leq m} \frac{h \hat{f}_p^+}{(1 + ikh)^{m-p+1}}. \quad (22)$$

Proof. The proof can be easily done via mathematical induction. First, the statement of the proposition is valid if $m = -M + 1$:

$$\widehat{R}_{-M+1}^+ = \frac{h \widehat{f}_{-M+1}^+}{1 + ikh}.$$

Having assumed that the proposition statement is valid for some $m > -M + 1$, let us prove that it is valid for $m + 1$. From Eq. (19) we have

$$\widehat{R}_{m+1}^+ = \frac{h \widehat{f}_{(m+1)-p+1}^+}{1 + ikh} = \sum_{p \leq m+1} \frac{h \widehat{f}_p^+}{(1 + ikh)^{(m+1)-p+1}}. \quad \square$$

Proposition 2. The solution of boundary value problem (20), (21) is given by

$$\widehat{R}_m^- = \frac{r_l \widehat{R}_M^+}{(1 + ikh)^{M-m}} + \sum_{p \geq m} \frac{h \widehat{f}_p^-}{(1 + ikh)^{1-m+p}}. \quad (23)$$

Here, the first term follows from (19) and integrating the homogeneous equation (20). The second term can be proved similarly to Proposition 1.

5. Discrete solution of AS problem with termination

Let us now consider a discrete solution of the AS problem. Using Theorem 1 we are able to find the AS source acting only at the boundary γ . It is to be noted that if we apply the theorem to Eq. (19), the operator L_h corresponds to the operator $ik + \frac{1}{h} \nabla$. In turn, if the result of the theorem is applied to Eq. (20), the operator L_h coincides with the operator $ik - \frac{1}{h} \Delta$. Consider now a partial case of the function $v^{(h)}$ in (12) such that for Eq. (19)

$$v_h = \begin{cases} \widehat{R}_{-1}^+, & \text{if } m = -1, \\ 0, & \text{else,} \end{cases} \quad (24)$$

while for Eq. (20)

$$v_h = \begin{cases} \widehat{R}_0^-, & \text{if } m = 0, \\ 0, & \text{else.} \end{cases} \quad (25)$$

Here, the values of \widehat{R}_{-1}^+ and \widehat{R}_0^- are to be taken from the measurements.

Then, the appropriate system of equations including the AS source terms becomes:

$$\left(ik + \frac{1}{h} \nabla \right) \widehat{R}_m^+ = \widehat{f}_m^+ + \widehat{g}_0^+ \delta_{m,0}, \quad (26)$$

$$\left(ik - \frac{1}{h} \Delta \right) \widehat{R}_m^- = \widehat{f}_m^- + \widehat{g}_0^- \delta_{m,0}, \quad m \in M^0. \quad (27)$$

Here, $\delta_{m,0} = 1$ if $m = 0$, otherwise $\delta_{m,0} = 0$. The AS source terms are given by

$$\begin{aligned}\hat{g}_0^+ &= \frac{\hat{R}_{-1}^+}{h}, \\ \hat{g}_0^- &= -\left(ik + \frac{1}{h}\right)\hat{R}_0^-. \end{aligned} \quad (28)$$

In our case, in order to substitute the measurement values, we are able to obtain them from exact solution (22), (23):

$$\hat{R}_0^- = \frac{r_l \hat{R}_M^+}{(1 + ikh)^M} + \sum_{p \geq 0} \frac{h \hat{f}_p^-}{(1 + ikh)^{1+p}}, \quad (29)$$

$$\hat{R}_{-1}^+ = \sum_{p \leq -1} \frac{h \hat{f}_p^+}{(1 + ikh)^{-p}}. \quad (30)$$

To demonstrate how the AS solution (28) performs, let us consider the two extreme cases of absolutely reflecting ($r_l = 1$) and absorbing ($r_l = 0$) walls on the right-hand side, respectively. Assume that in both cases the AS solutions coincide each other. In the case of the absorbing wall, this can be achieved if we include the following additional source term on the right-hand side of Eq. (20) to compensate for the absorption of the wave \hat{R}^+ :

$$\hat{f}_M^- = (1 - r_l) \frac{\hat{R}_M^+}{h}. \quad (31)$$

Then, in both cases we have the same \hat{R}_{-1}^+ and \hat{R}_0^- in the measurements, hence we obtain the same AS source terms. According to the main theorem, sources (28) completely provide the AS. Meanwhile, we can see, on one hand the AS filters the wave generated by \hat{f}_M^- (if $r_l = 0$) as the wave is related to the external source. On the other hand, formally the same AS remains the wave related to the reflection (if $r_l = 1$) and having the characteristics coinciding with the previous wave. Such a paradoxical situation, at first glance, can be explained via the analysis of Green's function.

The solution of problem (19)–(21) can be represented via Green's matrix function. For this purpose, the following two auxiliary problems are to be solved:

$$\begin{aligned}\left(ik + \frac{1}{h}\nabla\right)\hat{G}_{1,i|0}^{(1)} &= \delta_{i,0}, \\ \left(ik - \frac{1}{h}\Delta\right)\hat{G}_{2,i|0}^{(1)} &= 0, \quad m \in M^0, \end{aligned} \quad (32)$$

with the following boundary conditions:

$$\hat{G}_{1,-M|0}^{(1)} = 0, \quad \hat{G}_{2,M|0}^{(1)} = r_l \hat{G}_{1,M|0}^{(1)} \quad (33)$$

and

$$\begin{aligned} \left(ik + \frac{1}{h}\nabla\right)\widehat{G}_{1,i|0}^{(2)} &= 0, \\ \left(ik - \frac{1}{h}\Delta\right)\widehat{G}_{2,i|0}^{(2)} &= \delta_{i,0}, \quad m \in M^0, \end{aligned} \quad (34)$$

subject to the same kind of boundary conditions:

$$\widehat{G}_{1,-M|0}^{(2)} = 0, \quad \widehat{G}_{2,M|0}^{(2)} = r_l \widehat{G}_{1,M|0}^{(2)}. \quad (35)$$

The solutions of these problems are as follows:

$$\widehat{G}_{1,m|0}^{(1)} = \theta(m)\alpha_h^{-m-1}, \quad \widehat{G}_{2,m|0}^{(1)} = r_l \alpha_h^{-2M-1+m} \quad (36)$$

and

$$\widehat{G}_{1,m|0}^{(2)} = 0, \quad \widehat{G}_{2,m|0}^{(2)} = \theta(-m)\alpha_h^{m-1}. \quad (37)$$

Here, $m \in M^0$ and $\alpha_h = 1 + ikh$,

$$\theta(m) = \begin{cases} 0 & \text{if } m < 0, \\ 1 & \text{if } m \geq 0. \end{cases}$$

Thus, the appropriate Green's functions are given by

$$\underline{\widehat{G}}_{m|0}^{(1)} = \alpha_h^{-m-1} \begin{pmatrix} \theta(m) \\ r_l \alpha_h^{2(m-M)} \end{pmatrix}, \quad \underline{\widehat{G}}_{m|0}^{(2)} = \alpha_h^{m-1} \begin{pmatrix} 0 \\ \theta(-m) \end{pmatrix}. \quad (38)$$

Having introduced the following vectors

$$\widehat{W}_i = (\widehat{R}_i^+, \widehat{R}_i^-)^T, \quad (39)$$

$$\widehat{F}_i = (\widehat{f}_i^+, \widehat{f}_i^-)^T + \delta_{i,0}(\widehat{g}_0^+, \widehat{g}_0^-)^T \equiv (\widehat{F}^+, \widehat{F}^-)^T, \quad (40)$$

system (19), (20) can be written in the following form:

$$\widehat{L}_h \widehat{W}_h = \widehat{F}_h, \quad (41)$$

where \widehat{L}_h is the appropriate finite-difference operator of the system.

The solution of problem (19)–(21) can be represented via the following convolution operation:

$$\widehat{W} = \underline{\widehat{G}} * \widehat{F}, \quad (42)$$

where

$$\underline{\widehat{G}}_{m|0} = \begin{pmatrix} \theta(m)\alpha_h^{-(m+1)} & 0 \\ r_l \alpha_h^{-2M+m-1} & \theta(-m)\alpha_h^{m-1} \end{pmatrix}. \quad (43)$$

On the definition, this expression means as follows:

$$\widehat{W}_m = \sum_{-M}^M \widehat{G}_{m-p|p} \widehat{F}_p h. \quad (44)$$

In order to prove (42), the expression can be rewritten in the following equivalent form:

$$\widehat{W} = \widehat{F}^+ * \widehat{G}^{(1)} + \widehat{F}^- * \widehat{G}^{(2)}. \quad (45)$$

Then,

$$\begin{aligned} L_h \widehat{W}_h &= \widehat{F}_h^+ * L_h \widehat{G}_h^{(1)} + \widehat{F}_h^- * L_h \widehat{G}_h^{(2)} \\ &= \widehat{F}_h^+ * (\delta, 0)^T + \widehat{F}_h^- * (0, \delta)^T = \widehat{F}_h. \end{aligned} \quad (46)$$

Thus, the solution of problem (19)–(21) is given by

$$\begin{aligned} \widehat{R}^+ &= \widehat{G}_1^{(1)} * (\hat{f}^+ + \hat{g}_0^+ \delta_0) + \widehat{G}_1^{(2)} * (\hat{f}^- + \hat{g}_0^- \delta_0), \\ \widehat{R}^- &= \widehat{G}_2^{(1)} * (\hat{f}^+ + \hat{g}_0^+ \delta_0) + \widehat{G}_2^{(2)} * (\hat{f}^- + \hat{g}_0^- \delta_0). \end{aligned} \quad (47)$$

From (33) it immediately follows that the boundary conditions (21) are also satisfied.

If $m < 0$, for \widehat{R}^+ we have the following solution:

$$\widehat{R}_m^+ = \sum \theta(m-p) h \hat{f}_p^+ \alpha_h^{-m+p-1} = \sum_{p \leq m} h \hat{f}_p^+ \alpha_h^{-m+p-1}. \quad (48)$$

Thus, if $m < 0$, the additional term $\hat{g}_0^+ \delta_0$ in (26) provides the field coinciding with the solution without the effect of external noise (22). It is important to note that the result does not depend on the value of \hat{g}_0^+ . Thus, it can be set as $\hat{g}_0^+ = 0$. This means that no AS is required for the function \widehat{R}^+ . This conclusion also follows from the property of equation (19) and the R^+ -Riemann invariant. Nevertheless, under some conditions the contribution of \hat{g}_0^+ can be substantial for the AS of \widehat{R}^- . These conditions are discussed below.

If $m < 0$, the solution of problem (20), (21) is given by

$$\begin{aligned} \widehat{R}_m^- &= \sum_p r_l h \hat{f}_p^+ \alpha_h^{-2M-1+m+p} + \sum_{p \geq m} h \hat{f}_p^- \alpha_h^{-1+m-p} + h \hat{g}_0^- \alpha_h^{m-1} + r_l h \hat{g}_0^+ \alpha_h^{-2M-1+m} \\ &= r_l \widehat{R}_M^+ \alpha_h^{m-M} + r_l h \hat{g}_0^+ \alpha_h^{-2M-1+m} + \sum_{p \geq m} h \hat{f}_p^- \alpha_h^{m-1-p} + h \hat{g}_0^- \alpha_h^{m-1}. \end{aligned} \quad (49)$$

From the exact solution (23) we have

$$\widehat{R}_0^- = r_l \widehat{R}_M^+ \alpha_h^{-M} + \sum_{p \geq 0} h \hat{f}_p^- \alpha_h^{-1-p}. \quad (50)$$

Hence, if $m < 0$, from (28) it follows that

$$\widehat{R}_m^- = r_l \widehat{R}_{-1}^+ \alpha_h^{-2M-1+m} + \sum_{m \leq p < 0} h \widehat{f}_p^- \alpha_h^{m-1-p}. \quad (51)$$

Thus, solution (51) coincides with the solution of (19)–(21) without the “adverse” noise, i.e. $\widehat{f}_p^+ = 0$, $\widehat{f}_p^- = 0$ for $p \geq 0$. The first term in (51) is responsible for the “echo” of the “friendly” sound. Thus, the function of the additional sources in the AS solution provided by the primary theorem is not limited by noise elimination but can also include the restoration of the echo of the “friendly” sound. It is important to note for further discussion that in (51) only the echo of the “friendly” sound explicitly includes the reflection coefficient r_l .

Thus, even if a wave arising from the right-hand side of the shielded domain is the same and the AS source is the same, the filtering procedure might be different due to the influence of the boundary condition on Green’s function. It is important to underline here again that in the solution provided by the DPT the knowledge of Green’s function is not required. The current analysis only helps us to reveal how this solution performs. It is worth noting that the AS source terms remain the same if sonic waves are spread in a nonuniform medium, where Green’s function solution obtained above is not valid.

It is important to note that the AS solution requires neither the knowledge of the reflection coefficient r nor the knowledge of a noise source location. All the required information is included in the values \widehat{R}_{-1}^+ and \widehat{R}_0^- obtained from an experiment. This information can include the total contribution of both internal and external sources. Since the Riemann invariants do not carry a physical meaning, some physical values, e.g. the pressure and velocity, are to be measured. Taking into account the grid resolution requirement of $kh \ll 1$ all the measurements can be done at only one point.

It is worth noting that if $r_l > 0$ and \widehat{g}_0^+ is neglected, then the first term in (51) is absent and the AS filters the reflection of the “friendly” sound from the right-hand side as well. Thus, the AS source acts in the anti-phase to the wave incoming only if the echo of the “friendly” noise is not taken into account.

From the analysis of solution (49), the following important conclusions can be stated. If the sound echo is substantial and the AS is implemented by a point source, then it is uniquely determined. If the reflection of “friendly” sound is not important, the optimal AS corresponds to $\widehat{g}_0^+ = 0$. In this case, if the source \widehat{g}_0^+ exists, it operates in vain. From (30), it follows that if “friendly” sources are absent, \widehat{g}_0^+ automatically equals zero. In contrast to \widehat{g}_0^+ , the other source \widehat{g}_0^- is uniquely determined.

6. Continuous case

It is useful to compare the finite-difference solution against the continuous solution. In the continuous space, the Green’s matrix is as follows:

$$\widehat{G}(x, 0) = \begin{pmatrix} e^{-ikx}\theta(x) & 0 \\ r_l e^{-ikx} e^{-i2k(L-x)} & e^{ikx}\theta(-x) \end{pmatrix}. \quad (52)$$

It is possible to see that this continuous matrix corresponds to the limit of matrix (43) if $h \rightarrow 0$.

The vector of the AS point source term is represented by

$$\hat{q}_0(x) = \delta(x) (\hat{R}_0^+, -\hat{R}_0^-)^T$$

where $\delta(x)$ is the delta-function. Then, the exact solution of the continuous problem represented by $\hat{G} * \hat{F}$ is as follows.

If $x \leq 0$, the \hat{R}^+ -Riemann invariant is given by

$$\hat{R}^+(x) = e^{-ikx} \int_{-L}^x e^{ik\xi} \hat{f}^+(\xi) d\xi. \quad (53)$$

Hence,

$$\hat{R}^+(0) = \int_{-L}^0 e^{ik\xi} \hat{f}^+(\xi) d\xi. \quad (54)$$

If $x \leq 0$, \hat{R}^- -Riemann invariant is the following:

$$\begin{aligned} \hat{R}^-(x) &= r_l e^{ik(x-2L)} \int_{-L}^L e^{ik\xi} \hat{f}^+(\xi) d\xi + e^{ikx} \int_x^L e^{-ik\xi} \hat{f}^-(\xi) d\xi + r_l e^{ik(x-2L)} \hat{R}_0^+ - e^{ikx} \hat{R}_0^- \\ &= r_l e^{ik(x-L)} \hat{R}_L^+ + e^{ikx} \int_x^L e^{-ik\xi} \hat{f}^-(\xi) d\xi + r_l e^{ik(x-2L)} \hat{R}_0^+ - e^{ikx} \hat{R}_0^-. \end{aligned} \quad (55)$$

Taking into account

$$\hat{R}_0^- = r_l e^{-ikL} \hat{R}_L^+ + \int_0^L e^{-ik\xi} \hat{f}^-(\xi) d\xi, \quad (56)$$

we have

$$\hat{R}^-(x) = r_l e^{ik(x-2L)} \hat{R}_0^+ + e^{ikx} \int_x^L e^{-ik\xi} \hat{f}^-(\xi) d\xi. \quad (57)$$

Let us introduce the transition matrix A from the Riemann invariants to the original variables such that

$$(\hat{R}^+, \hat{R}^-)^T = A(\hat{p}, \hat{u})^T.$$

Then,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and Green's function in the original variables is given by

$$\widehat{G} = A^{-1} \widehat{\underline{G}} A. \quad (58)$$

Ultimately, Green's function is as follows:

$$\widehat{G}(x, 0) = \frac{1}{2} \begin{pmatrix} \Theta_+ + \Psi & \Theta_- + \Psi \\ \Theta_- - \Psi & \Theta_+ - \Psi \end{pmatrix}, \quad (59)$$

where $\Theta_{\pm}(x) = e^{-ikx}\theta(x) \pm e^{ikx}\theta(-x)$, $\Psi(x, L) = r_l e^{ik(x-2L)}$.

The source term in the original variables is given by

$$\hat{q}_0(x) = A^{-1} \hat{g}_0(x) = \delta(x)(\hat{u}_0, \hat{p}_0)^T, \quad (60)$$

while under the requirement of approximation $kh \ll 1$, the finite-difference AS source term is as follows:

$$\hat{q}_0^{(h)} = A^{-1} \hat{g}_0^{(h)} = \frac{1}{h}(\hat{u}_0, \hat{p}_0)^T. \quad (61)$$

It is worth representing the source term (61) via the Riemann invariants $\widehat{R}^+ = \frac{\hat{p}}{\rho c} + \hat{u}$ and $\widehat{R}^- = \frac{\hat{p}}{\rho c} - \hat{u}$:

$$\hat{q}_0^{(h)} = A^{-1} \hat{g}_0^{(h)} = \frac{1}{2h}(\widehat{R}_0^+ - \widehat{R}_0^-, \widehat{R}_0^+ + \widehat{R}_0^-)^T. \quad (62)$$

If the reflection is not important, the AS source term (60) is not minimal in L_2 . In order to show this, the energetic norm of the source can be represented via the Riemann invariants:

$$\|\hat{q}_0^{(h)}\|_2^2 = \frac{1}{4h^2}(|\widehat{R}_0^+ + \widehat{R}_0^-|^2 + |\widehat{R}_0^+ - \widehat{R}_0^-|^2) \geq \frac{1}{2h^2}|\widehat{R}_0^-|^2. \quad (63)$$

It is easy to see that the energetically optimal solution corresponds to $\widehat{R}_0^+ = 0$. This solution also has the minimal norm in L_1 . This immediately follows from the next simple inequality:

$$\frac{1}{2}|\widehat{R}^- + \widehat{R}^+| + \frac{1}{2}|\widehat{R}^- - \widehat{R}^+| \geq |\widehat{R}^-|. \quad (64)$$

Thus, the minimum is reached at $\widehat{R}^+ = 0$ in both L_1 and L_2 . In contrast to [12–14], we consider the optimization having assumed *a priori* that the AS source term is situated only at boundary γ^- .

If $x < 0$, the contribution to the solution in the original variables made by the AS is given by

$$\widehat{W}_c = \frac{e^{ikx}}{2}(\widehat{R}_0^- - r_l e^{-i2kL} \widehat{R}_0^+)(-1, 1)^T. \quad (65)$$

This field provides the AS if we have the delta-function source (60). In reality we are not able to implement such a source. Instead, we consider the finite-difference solution (61) and assume that the source has a physical extent of Δ :

$$\hat{q}(x) = \frac{1}{h} \theta(x + \Delta/2) [1 - \theta(x - \Delta/2)] (\hat{u}_0, \hat{p}_0)^T. \quad (66)$$

This source term generates the following field:

$$\widehat{W}_h = G * \hat{q} = \frac{2\widehat{W}_c}{kh} \sin \frac{k\Delta}{2}. \quad (67)$$

In order to generate the same field \widehat{W}_c , the parameter h is taken to be approximately equal to the thickness of the source Δ :

$$h = \frac{2}{k} \sin \frac{k\Delta}{2} \approx \Delta, \quad (68)$$

and the thickness of the source must be small enough to satisfy the requirement $k\Delta \ll 1$. Relation (68) seems to be natural, since h should be as small as possible on one hand and not less than the source thickness on the other hand. The latter requirement follows from the assumption of the point AS source.

In the original variables, the governing equations for the Fourier spectrum including the AS are as follows:

$$\begin{aligned} ik\hat{p} + \hat{u}_x &= \hat{f}_p + \hat{u}_0\delta(x), \\ ik\hat{u} + \hat{p}_x &= \hat{f}_u + \hat{p}_0\delta(x). \end{aligned} \quad (69)$$

For the original physical time-dependent variables we have

$$\begin{aligned} p_t + \rho c^2 u_x &= \rho c^2 q_{\text{vol}} + f_p, \\ u_t + \frac{p_x}{\rho} &= \frac{b_{\text{vol}}}{\rho} + f_u, \end{aligned} \quad (70)$$

where $q_{\text{vol}}(x) = u_0\delta(x)$, $b_{\text{vol}}(x) = p_0\delta(x)$, $u_0 = \hat{u}_0 e^{i\omega t}$, $p_0 = \rho c \hat{p}_0 e^{i\omega t}$.

Here, q_{vol} and b_{vol} are the volumetric monopole and dipole sources, respectively. They appropriately alter the acoustic balance of mass and momentum in the system [12].

The wave generated by the AS in $x < 0$ is as follows:

$$W_c = ((\rho c q_{\text{vol}} - b_{\text{vol}}) + r_l e^{-i2kL} (\rho c q_{\text{vol}} + b_{\text{vol}})) \frac{e^{ikx}}{2} (1, -1/(\rho c))^T. \quad (71)$$

If the reflection of “friendly” sound is not substantial, the AS solution minimal in L_1 and L_2 is represented by the following sources:

$$q_{\text{vol|opt}} = -(p_0 - \rho c u_0) \frac{\delta(x)}{2\rho c}, \quad (72)$$

$$b_{\text{vol|opt}} = (p_0 - \rho c u_0) \frac{\delta(x)}{2}. \quad (73)$$

This immediately follows from (62). In this case only the combination $\rho c q_{\text{vol}} - b_{\text{vol}}$ of the sources q_{vol} and b_{vol} is substantial. Therefore, some simplified interpretations are also available, for instance, the following source terms:

$$q_{\text{vol|opt}} = -(p_0 - \rho c u_0) \frac{\delta(x)}{\rho c}, \quad b_{\text{vol|opt}} = 0. \quad (74)$$

If both the “friendly” sound and reflection on the left-hand side are absent, then the measurement of either the pressure or the velocity can be omitted. In the latter case, the source terms can be represented in the following form:

$$q_{\text{vol|opt}} = -\frac{2p_0}{\rho c} \delta(x), \quad b_{\text{vol|opt}} = 0. \quad (75)$$

Considering the solution generated by the AS (71), it is easy to see that the solutions for p and u can be distinguished from each other only by the sign. Thus, if we are able to generate the AS solution for the pressure, then we are able to obtain it for the velocity. In other words, the capability to suppress the adverse solution for one of the dependent variables is enough to resolve the total AS task.

7. Duct with both-side termination

Let us now consider the more general formulation of the problem including a left-hand duct termination. As will be shown later, altering the boundary condition (21) on the left-hand side $m = -M$ does not change the principal conclusions given above, if the solution remains unique.

In the general case, we have the following boundary conditions:

$$\widehat{R}_{-M}^+ = r_{-l} \widehat{R}_{-M}^-, \quad \widehat{R}_M^- = r_l \widehat{R}_M^+, \quad (76)$$

where r_{-l} is the reflection coefficient on the left-hand side.

It leads to the following boundary conditions for Green's function:

$$\widehat{G}_{1,-M|0}^{(1)} = r_{-l} \widehat{G}_{2,-M|0}^{(1)}, \quad \widehat{G}_{2,M|0}^{(1)} = r_l \widehat{G}_{1,M|0}^{(1)}, \quad (77)$$

$$\widehat{G}_{1,-M|0}^{(2)} = r_{-l} \widehat{G}_{2,-M|0}^{(2)}, \quad \widehat{G}_{2,M|0}^{(2)} = r_l \widehat{G}_{1,M|0}^{(2)}, \quad (78)$$

instead of (33) and (35), respectively.

Then, it is possible to show that Green's matrix function is given by

$$\widehat{G}_{m|0} = \begin{pmatrix} (A^+ + \theta(m))\alpha_h^{-m-1} & B_M^+ \alpha_h^{-m-1} \\ A_{-M}^- \alpha_h^{m-1} & (B^- + \theta(-m))\alpha_h^{m-1} \end{pmatrix}, \quad (79)$$

where

$$A^+ = \frac{r_l r_{-l}}{\Delta_h}, \quad A_{-M}^- = \frac{r_l \alpha_h^{2M}}{\Delta_h}, \quad B_M^+ = \frac{r_{-l} \alpha_h^{2M}}{\Delta_h},$$

$$B^- = \frac{r_l r_{-l}}{\Delta_h}, \quad \Delta_h = \alpha_h^{4M} - r_l r_{-l}.$$

From (79), it follows that if $\Delta_h = 0$, the solution does not exist. This case corresponds to a resonance.

In the continuous case the Green's matrix function is as follows:

$$\widehat{G}(x, 0) = \begin{pmatrix} (A^+ + \theta(x))e^{-ikx} & B^+(L)e^{-ikx} \\ A^-(-L)e^{ikx} & (B^- + \theta(-x))e^{ikx} \end{pmatrix}, \quad (80)$$

where

$$A^+ = \frac{r_l r_{-l}}{\Delta}, \quad A^-(-L) = \frac{r_l e^{i2kL}}{\Delta}, \quad B^+(L) = \frac{r_{-l} e^{i2kL}}{\Delta},$$

$$B^- = \frac{r_l r_{-l}}{\Delta}, \quad \Delta = e^{i4kL} - r_l r_{-l}.$$

In the continuous case the resonance condition then corresponds to $r_l r_{-l} = e^{i4kL}$. From the discussion given above, it follows that the implementation of the AS is based on the finite-difference solution, rather than on the continuous solution and the condition $\Delta_h = 0$ can be avoided by altering the parameter h .

In the general case the measurement data at $x = 0$ corresponds to

$$\widehat{R}^+(0) = \int_{-L}^0 e^{ik\xi} \widehat{f}^+(\xi) d\xi + \frac{r_l r_{-l}}{\Delta} \int_{-L}^L e^{ik\xi} \widehat{f}^+(\xi) d\xi$$

$$+ \frac{r_{-l}}{\Delta} e^{i2kL} \int_{-L}^L e^{-ik\xi} \widehat{f}^-(\xi) d\xi. \quad (81)$$

Here, whereas the first term obviously represents the immediate influence of “friendly” sources and it coincides with (54), there appear two additional terms describing the waves generated by total sources in the duct. The second term in (81) represents the waves reflected from the right-hand boundary and then the left-hand boundary; the third term is responsible for the waves reflected from the left-hand side.

Thus, in this case, even if there are no sources of “friendly” sound, the Riemann invariant $\widehat{R}^+(0)$ is not zero. It requires both sources q_{vol} and b_{vol} .

Green's function is represented by the following matrix-function:

$$\widehat{G}(x, 0) = \frac{r_{-l} r_l}{\Delta} \begin{pmatrix} \cos kx & -i \sin kx \\ -i \sin kx & \cos kx \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \Theta_+ & \Theta_- \\ \Theta_- & \Theta_+ \end{pmatrix}$$

$$+ \frac{e^{ik2L}}{2\Delta} \left[r_{-l} e^{-ikx} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + r_l e^{ikx} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right]. \quad (82)$$

Then, in the general case the wave generated by the AS in $x < 0$ is represented as follows:

$$\begin{aligned}
 W_c = & \left((\rho c q_{\text{vol}} - b_{\text{vol}}) + \frac{r_l}{\Delta} e^{i2kL} (\rho c q_{\text{vol}} + b_{\text{vol}}) \right) \frac{e^{ikx}}{2} (1, -1/(\rho c))^T \\
 & + \frac{r_{-l}}{\Delta} e^{i2kL} (\rho c q_{\text{vol}} - b_{\text{vol}}) \frac{e^{ikx}}{2} (1, 1/(\rho c))^T \\
 & + \frac{r_{-l} r_l}{\Delta} \begin{pmatrix} \rho c q_{\text{vol}} \cos kx - i b_{\text{vol}} \sin kx \\ -i q_{\text{vol}} \sin kx + b_{\text{vol}}/(\rho c) \cos kx \end{pmatrix}. \quad (83)
 \end{aligned}$$

It is easy to see that if $r_l = 0$, then the AS solution is non-unique and the optimal solutions (72), (74) are applicable.

8. Conclusion

Now we are able to arrive at the following conclusions. The mechanism of the general formal solution of the AS problem has been detailed for the case of monochromatic wave propagation in a duct with an end termination. The one-layer AS solution has been obtained; the solution only requires the measurement results for the total acoustic field (both “friendly” and “adverse”) at the boundary of the shielded area. It requires neither the knowledge of the medium where the acoustic field is propagated nor the boundary reflection coefficient values. The correspondence between the finite-difference solution and continuous solution is shown for the case of a uniform medium. In practice, the AS source can be implemented via an extended uniform source corresponding to a discrete solution containing a parameter h . It is shown that the optimal value of h corresponds to the thickness of the source which is required to be much less than the wave length. For a point AS source, if either the external area is not acoustically terminated or the reflection of “friendly” sound is not substantial, the AS is not unique and hence there is room for optimization. The optimal solution has been found. These conclusions are generally anticipated to be important for future application of the technique to spatial problems.

References

- [1] J.C. Burgess, Active adaptive sound control in a duct: A computer simulation, *J. Acoust. Soc. Amer.* 70 (1981) 715–726.
- [2] R.H. Cabell, C.R. Fuller, Active control of periodic disturbances using principal component LMS: Theory and experiment, in: 3rd AST/HSR Interior Noise Workshop, Part I: Sessions A, B, and C, NASA Langley Research Center, Hampton, VA, 1998.
- [3] S.J. Elliott, I.M. Stothers, P.A. Nelson, A multiple error LMS algorithm and its application to the active control of sound and vibration, in: *IEEE Trans., Acoustics, Speech and Signal Processing ASSP-35*, 1987, pp. 1423–1434.
- [4] M.V. Fedoryuk, An unsteady problem of active noise suppression, *Acoustic J.* 22 (1976) 439–443 (in Russian).
- [5] C.R. Fuller, P.A. Nelson, S.J. Elliott, *Active Control of Vibration*, Academic Press, San Diego, CA, 1996.
- [6] V. Isakov, *Inverse Source Problems*, Math. Surveys Monogr., vol. 34, Amer. Math. Soc., Providence, RI, 1990.
- [7] M.J. Jessel, Sur les absorbeurs actifs, in: *Proceedings 6th ICA*, Tokyo, 1968, p. 82, Paper F-5-6.
- [8] M.J. Jessel, Some evidences for a general theory of active noise sound absorption, in: *Proceedings of Inter-Noise 79*, Warsaw, 1979, pp. 169–174.
- [9] R.K. Kincaid, K. Laba, Reactive tabu search and sensor selection in active structural control problems, *J. Heuristics* 4 (1998) 199–220.
- [10] R.K. Kincaid, S.L. Padula, D.L. Palumbo, Optimal sensor/actuator locations for active structural acoustic control, AIAA Paper 98-1865, in: *Proceedings of the 39th AIAA/ASME/ASCE/AHS/ASC Structures, Dynamics and Materials Conference*, Long Beach, CA, 1998.

- [11] J. Lončarić, V.S. Ryaben'kii, S.V. Tsynkov, Active shielding and control of noise, *SIAM J. Appl. Math.* 62 (2001) 563–596.
- [12] J. Lončarić, S.V. Tsynkov, Optimization of acoustic source strength in the problems of active noise control, *SIAM J. Appl. Math.* 63 (2003) 1141–1183.
- [13] J. Lončarić, S.V. Tsynkov, Optimization of power in the problem of active control of sound, *Math. Comput. Simulation* 65 (2004) 323–335.
- [14] J. Lončarić, S.V. Tsynkov, Quadratic optimization in the problems of active control of sound, *Appl. Numer. Math.* 52 (2005) 381–400.
- [15] G.D. Malyuzhinets, An unsteady diffraction problem for the wave equation with compactly supported right-hand side, in: *Proceedings of the Acoustics Inst., USSR Ac. Sci., Moscow*, vol. 15, 1971, pp. 124–139 (in Russian).
- [16] G. Mangiante, The JMC method for 3D active sound absorption: A numerical simulation, *Noise Control Engineering J.* 41 (2) (1993) 339–345.
- [17] P.A. Nelson, A.R.D. Curtis, S.J. Elliott, A.J. Bullmore, The minimum power output of free field point sources and the active control of sound, *J. Sound Vibration* 116 (1987) 397–414.
- [18] P.A. Nelson, S.J. Elliott, *Active Control of Sound*, Academic Press, San Diego, CA, 1992.
- [19] J. Piraux, B. Nayroles, A theoretical model for active noise attenuation in three-dimensional space, in: *Proceedings of Inter-Noise 80*, Miami, 1980, pp. 703–706.
- [20] V.S. Ryaben'kii, A difference shielding problem, *Funct. Anal. Appl.* 29 (1995) 70–71.
- [21] V.S. Ryaben'kii, *Method of Difference Potentials and Its Applications*, Springer-Verlag, Berlin, 2002.
- [22] V.S. Ryaben'kii, S.V. Utyuzhnikov, Differential and finite-difference problems of active shielding, *Appl. Numer. Math.* (2006), in press.
- [23] O. Tochi, S. Veres, *Active Sound and Vibration Control. Theory and Applications*, The Institution of Electrical Engineers, New York, 2002.
- [24] S.V. Tsynkov, On the definition of surface potentials for finite-difference operators, *J. Sci. Comput.* 18 (2003) 155–189.
- [25] S. Uosukainen, V. Välimäki, *JMC Actuators and Their Applications in Active Attenuation of Noise in Ducts*, VTT Publications, vol. 341, VTT Building Technology, ESPOO, 1998, 100 pp.
- [26] B. Widrow, D. Schur, S. Sohareff, On adaptive inverse control, in: *Proceedings of the Asilomar Conference on Circuits, Systems and Computers*, Pacific Grove, CA, 1981, pp. 185–195.
- [27] S.E. Wright, B. Vuksanovic, Active control of environment noise, II: Non-compact acoustic sources, *J. Sound Vibration* 202 (1997) 313–359.